Numerical Hydrodynamics
Part 2

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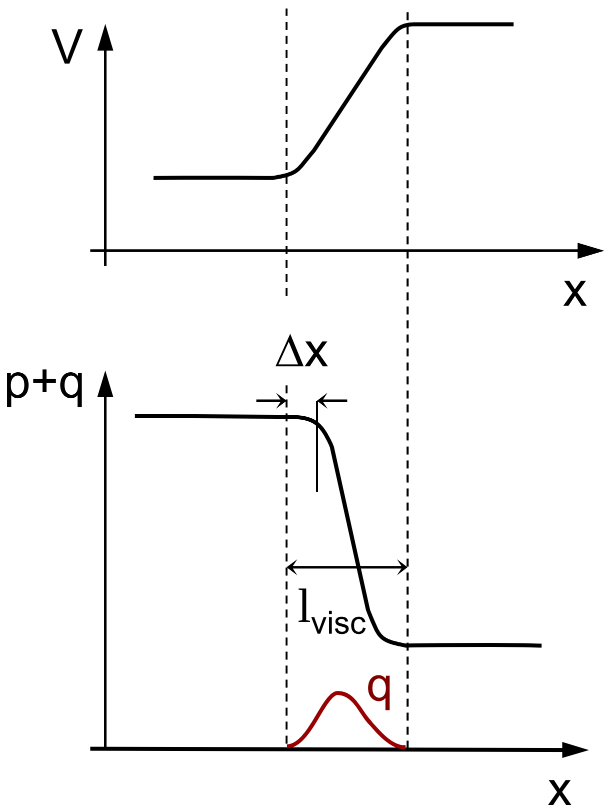
Outline

- Artificial viscosity
- Operator Splitting
- Adaptive Grids
- Implicit computations for radiation hydrodynamics
- Conclusions

Artificial Viscosity I



- Artificial viscosity sometimes needed to broaden shock fronts over a few grid cells to ensure smooth solutions
- Von Neumann & Richtmyer (1950): artificial pressure q in 1D-flows in the vicinity of shock fronts, i.e. du/dx <0

$$q = \frac{(\rho\, l_{\rm visc})^2}{V}\left(\frac{\partial V}{\partial t}\right)^2 = \rho\, l_{\rm visc}^2 \left(\frac{du}{dx}\right)^2$$

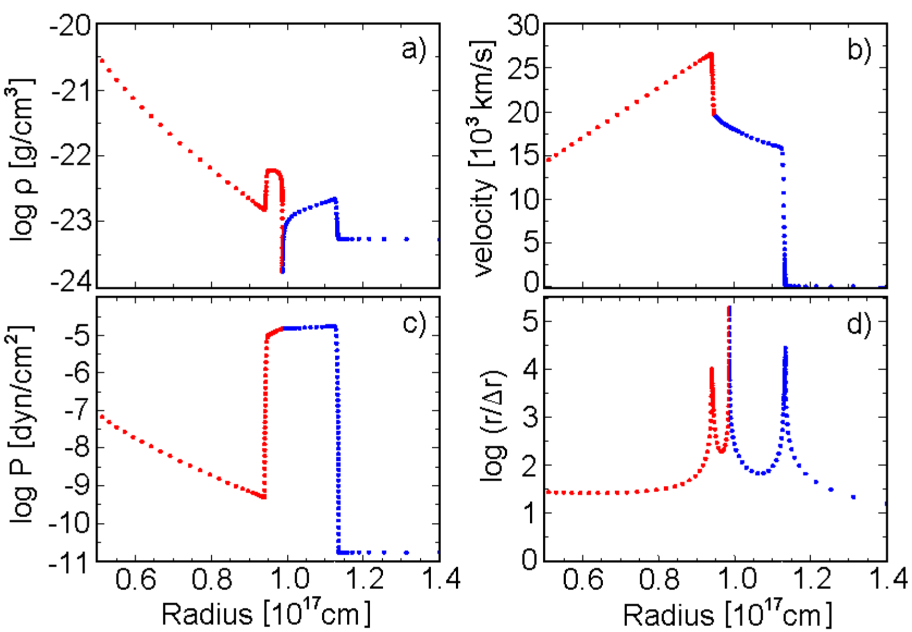
$$l_{\rm visc} \simeq 2 \ldots 3\, \Delta x$$

- Additional local dissipation term but Rankine-Hugoniot conditions are valid if integrating over $l > l_{visc}$

Artificial Viscosity II

$$\mu_{\mathrm{Q}} = -q_1 l_{\mathrm{visc}} c_s + q_2^2 l_{\mathrm{visc}}^2 \min(\nabla \cdot u, 0)$$

$$\begin{aligned}
\mathbf{Q} &= \mu_{\mathrm{Q}} \left[(\nabla \mathbf{u}) - \frac{1}{3} \nabla \cdot u \, \mathbf{I} \right] \\
u_{\mathrm{Q}} &= -\nabla \cdot \mathbf{Q} \\
\epsilon_{\mathrm{Q}} &= -\frac{1}{\rho} \mathbf{Q} \cdot (\nabla \mathbf{u}),
\end{aligned}$$



- Viscosity only, if flow is convergent, $\nabla \cdot u < 0$
- No viscous terms for homologous contractions
- Physically motivated Tensor formulation due to Tscharnuter and Winkler (1979)
- Q viscous pressure, u_Q viscous forces, ε_Q viscous energy dissipation
- 1^{st} and 2^{nd} order terms
- Length scales l_{visc} should be smaller than physical scales, usable on adaptive grids

Diffusion Equation in 2D

$$\frac{\partial u}{\partial t} = \sigma \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

- Alternating-Direction Method **(ADI)** developed by Peaceman and Rachford (1955)

$$u^{(n+1/2)} - u^{(n)} = \frac{1}{2} h \left(\delta_x^2 u^{(n+1/2)} + \delta_y^2 u^{(n)} \right)$$

$$u^{(n+1)} - u^{(n+1/2)} = \frac{1}{2} h \left(\delta_x^2 u^{(n+1/2)} + \delta_y^2 u^{(n+1)} \right)$$

- Simple and unconditionally stable by two successive time steps, 2nd order

$$h = \frac{\sigma \Delta t}{(\Delta x)^2}$$

$$\delta_x^2 u = u_{j-1,k} - 2u_{j,k} + u_{j+1,k}$$

$$\delta_y^2 u = u_{j,k-1} - 2u_{j,k} + u_{j,k+1}$$

$$\varepsilon = \mathcal{O}(\Delta t)^2 + \mathcal{O}(\Delta x)^2$$

Diffusion Equation in 3D

$$\frac{\partial u}{\partial t} = \sigma \, \Delta u = \sigma \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

- In 3D this simple version does not work, because one gets a cyclic shifting of the implicit term
- Douglas and Gunn (1964): Sequence of 3 approximate solutions

$$u^* - u^{(n)} = \frac{h}{2} \left[\delta_x^2 (u^* + u^{(n)}) + \delta_y^2 (2u^{(n)}) + \delta_z^2 (2u^{(n)}) \right]$$

$$u^{**} - u^{(n)} = \frac{h}{2} \left[\delta_x^2 (u^* + u^{(n)}) + \delta_y^2 (u^{**} + u^{(n)}) + \delta_z^2 (2u^{(n)}) \right]$$

$$u^{(n+1)} - u^{(n)} = \frac{h}{2} \left[\delta_x^2 (u^* + u^{(n)}) + \delta_y^2 (u^{**} + u^{(n)}) + \delta_z^2 (u^{(n+1)} + u^{(n)}) \right]$$

$$\varepsilon = \mathcal{O}(\Delta t)^2 + \mathcal{O}(\Delta x)^2$$

Elliptical Poisson Equation

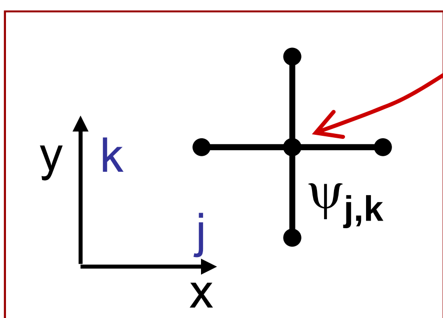
$$\frac{\partial \psi}{\partial t} = \Delta\psi - 4\pi G\rho \quad \text{with} \quad \frac{\partial \psi}{\partial t} \rightarrow 0$$

$$\vec{g} = -\nabla\psi$$

- Poisson Equation for computing the gravitational potential ψ, elliptical equation
- Stationary solution of a pseudo diffusion equation for ψ
- Direct solution through matrix inversion, here: 2D version

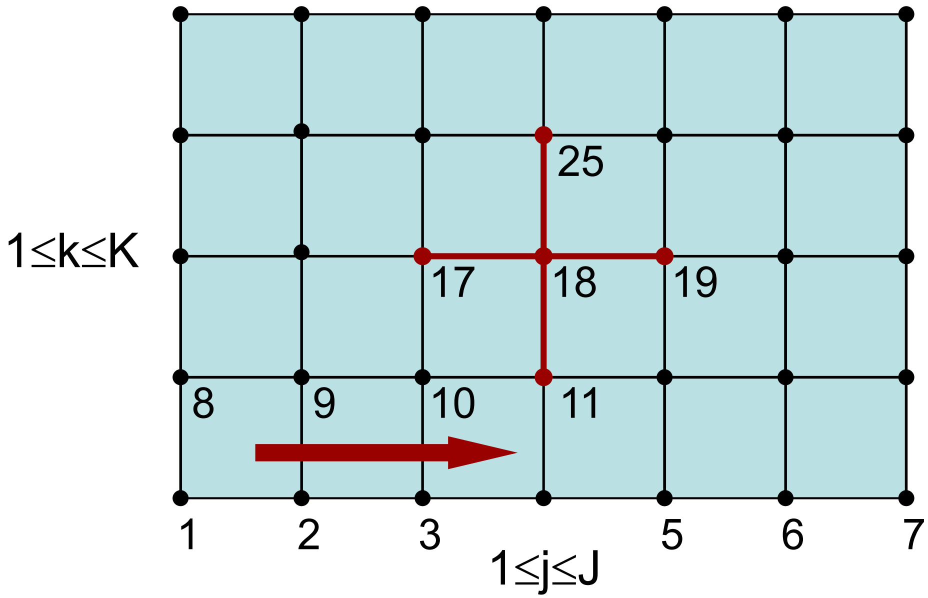
$$\frac{\psi_{j-1,k} - 2\psi_{j,k} + \psi_{j+1,k}}{\Delta x^2} + \frac{\psi_{j,k-1} - 2\psi_{j,k} + \psi_{j,k+1}}{\Delta y^2} - 4\pi G\rho_{j,k} = 0$$

$$\Rightarrow \psi_{j-1,k} + \psi_{j,k-1} - 4\psi_{j,k} + \psi_{j+1,k} + \psi_{j,k+1} = 4\pi G\rho_{j,k}\Delta x^2 \; (= d_{j,k})$$



- Stencil is mapped on (Jacobi)-matrix
- Reduction of band width to reduce operations on the matrix inversion

Matrix structure



$$A = \begin{pmatrix} -4 & 1 & 0\,0 \cdots 0\,1\,0\,0 \cdots & & \cdots \\ 1 & -4 & 1\,0 \cdots 0\,0\,1\,0 \cdots & & \cdots \\ & \ddots & \ddots & \ddots & \\ \cdots 0\,1\,0 \cdots 0\,1 & -4 & 1\,0 \cdots 0\,0\,1\,0 \cdots & & \\ \cdots & & & & \cdots \end{pmatrix}$$

$$A\vec{\psi} = \vec{d} \qquad \vec{\psi} = A^{-1}\vec{d}$$

- All variables are connected
- 2D-array of indices (j,k) has to be mapped one-to-one onto a 1D-array, which determines non-zero-elements within the matrix A
- Simple sparse matrix, special structure, specific algorithms have been developed for these matrices

Dimensional Splitting (I)

$$\frac{\partial q}{\partial t} = \mathcal{A}\, q$$

- Extension of 1D methods to more spatial dimensions, e.g. ADI
- Fractional steps by splitting into 1D subproblems

$$\begin{aligned} q(x,t) &= q(x,0) + t\frac{\partial q}{\partial t}|_{x,0} + \frac{1}{2}t^2\frac{\partial^2 q}{\partial t^2}|_{x,0} + \cdots \\ &= q(x,0) + t\mathcal{A}q(x,0) + \frac{t^2}{2!}\mathcal{A}^2 q(x,0) + \cdots = q(x,0)e^{t\mathcal{A}} \end{aligned}$$

$$e^{t(\mathcal{A}+\mathcal{B})} = 1 + t(\mathcal{A}+\mathcal{B}) + \frac{t^2}{2}(\mathcal{A}^2 + \mathcal{A}\mathcal{B} + \mathcal{B}\mathcal{A} + \mathcal{B}^2) + \mathcal{O}(t^3)$$

$$\begin{aligned} e^{t\mathcal{A}}e^{t\mathcal{B}} &= \left(1 + t\mathcal{A} + \frac{t^2}{2!}\mathcal{A}^2 + \mathcal{O}(t^3)\right)\left(1 + t\mathcal{B} + \frac{t^2}{2!}\mathcal{B}^2 + \mathcal{O}(t^3)\right) \\ &= 1 + t(\mathcal{A}+\mathcal{B}) + \frac{t^2}{2}(\mathcal{A}^2 + 2\mathcal{A}\mathcal{B} + \mathcal{B}^2) + \mathcal{O}(t^3) \end{aligned}$$

Dimensional Splitting (II)

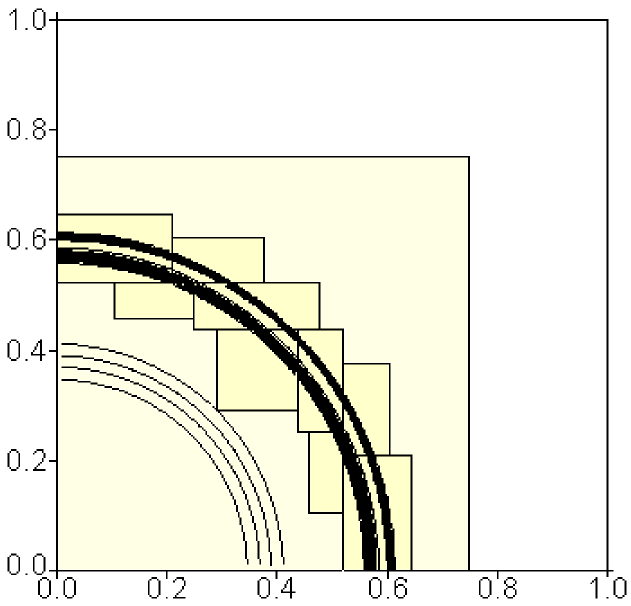
$$\frac{\partial q}{\partial t} = (\mathcal{A} + \mathcal{B})q$$

- Splitting error if fractional operators do not commute, q** is the approximate solution

$$q(x, \Delta t) - q^{**}(x, \Delta t) = \frac{1}{2}(\Delta t)^2(\mathcal{B}\mathcal{A} - \mathcal{A}\mathcal{B})q(x, 0) + \mathcal{O}((\Delta t)^3)$$

- No unique way (or mathematical theory) how to split the operator
- Various methods (or recipes), easy to implement
- Testing on 1-D problems
- Very efficient for implicit methods, significant reduction of operations

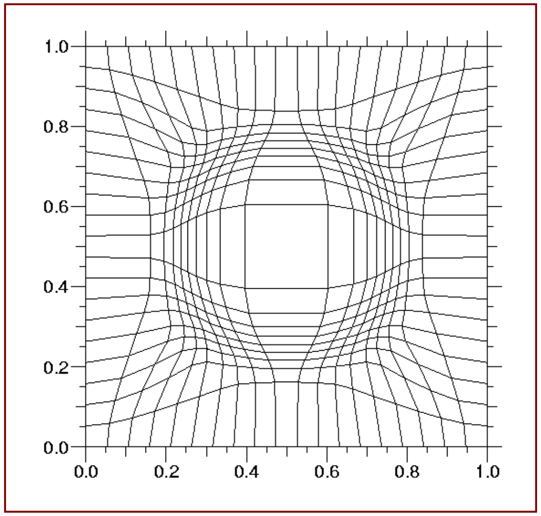
Adaptive Mesh Refinement (AMR)

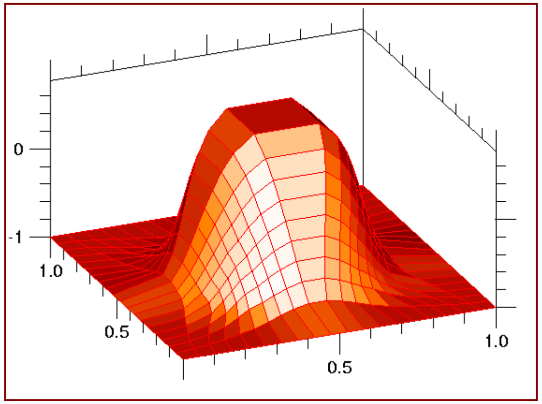


AMR with 4 levels

- Necessary for 2D or 3D computations, complex geometry
- Non-uniform grid, either boundary fitted, fixed or moving in space
- Wide range of spatial and temporal scales
- Refinement essential for shock and moving fronts, front or shock tracking
- AMR: refinement of individual cells, several levels of refinement possible
- Solutions are obtained with different numerical time steps depending on grid size

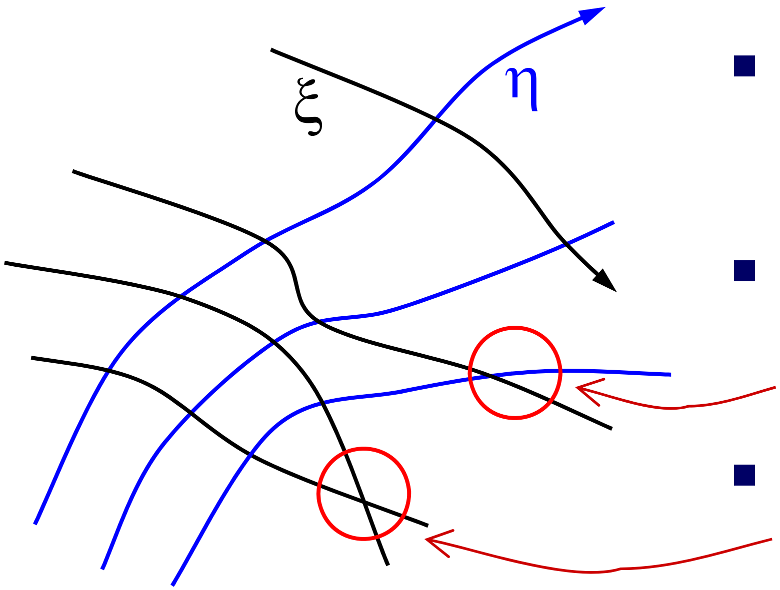
Moving Grids





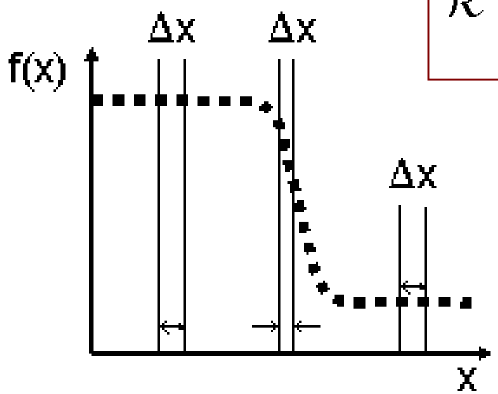
- Basic idea: Fixed number of grid points (cells) can move during the computation to increase the resolution at steep gradients
- The physical position of (x_j, y_k) is mapped on the discrete index space (j,k) at every time step
- Location (position) is a new variable
- Additional equation, so-called grid-equation controls the motion of the grid points
- Eulerian frame: $$\mathbf{u}^{\mathrm{grid}} = \frac{d\mathbf{x}}{dt} = 0$$
- Lagrangian frame $$\mathbf{u}^{\mathrm{grid}} = \mathbf{u}_{\mathrm{fluid}}$$

Grid properties



- Grid should trace fronts automatically to resolve steep features
- Refinement of cells or growth of cell sizes not too fast, not more than a factor of 2 in length (see AMR methods)
- Shape of cell has to be controlled, coordinate lines almost orthogonal, avoid skewness, e.g. no needles
- Grid cell should remain regular, i.e. no folding or twisting with negative volumes, simpler: monotonicity in 1D-problems
- Robust and only few parameters to control behaviour

1D - Adaptive Grid I



$$\mathcal{R} \propto \sqrt{1 + \left(\frac{df}{dx}\right)^2}$$

$$n_j = \frac{\mathcal{X}_j}{x_j - x_{j+1}}$$

$$\mathcal{R}_j \propto \sqrt{1 + \sum_{m=1}^{M} \left(\frac{\mathcal{X}_j}{F_{j,m}} \frac{f_{j,m} - f_{j+1,m}}{x_j - x_{j+1}}\right)^2} = \sqrt{1 + n_j^2 \sum_{m=1}^{M} g_m \left(\frac{f_{j,m} - f_{j+1,m}}{F_{j,m}}\right)^2}$$

- Fixed number of N grid points: r_j, $1 \le j \le N$
- Grid points must remain monotonic: $r_j < r_{j+1}$
- Grid positions are rearranged at every time-step
- Additional grid equation is solved together with the physical equations
- Define point concentration n_j
- What do you want to resolve (density, temperature, opacity, kinetic energy, …) → desired resolution

1D - Adaptive Grid II

$$\frac{\alpha_g}{\alpha_g+1} \le \frac{n_j}{n_{j+1}} \le \frac{\alpha_g+1}{\alpha_g}$$

$$n_j \propto \sum_l \mathcal{R}_l \left(\frac{\alpha_g}{\alpha_g+1}\right)^{|j-l|}$$

- Grid points are basically distributed along the arc-length of a physical quantity
- Spatial and temporal smoothing applied to the desired resolution

$$\hat{n}_j = n_j - \alpha_g(\alpha_g+1)\,(n_{j-1} - 2n_j + n_{j+1}) \propto \mathcal{R}_j$$

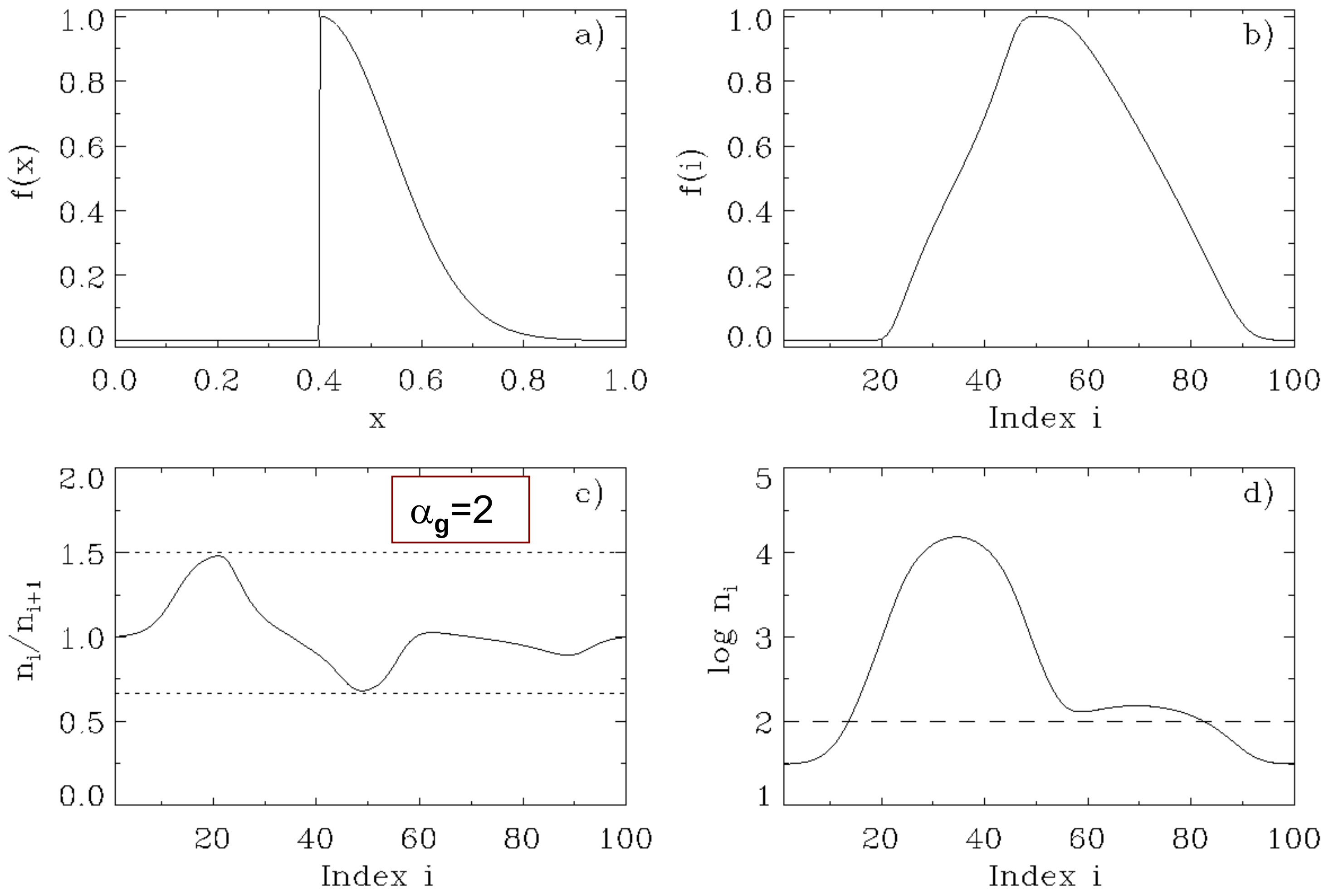
$$\mathcal{R}(t) = \int_0^\infty \mathcal{R}(t-t')\exp\left(\frac{-t'}{\tau_g}\right)\frac{dt'}{\tau_g}$$

$$\tilde{n}_j^{(n+1)} = \hat{n}_j^{(n+1)} + \frac{\tau_g}{\Delta t}\left(\hat{n}_j^{(n+1)} - \hat{n}_j^{(n)}\right)$$

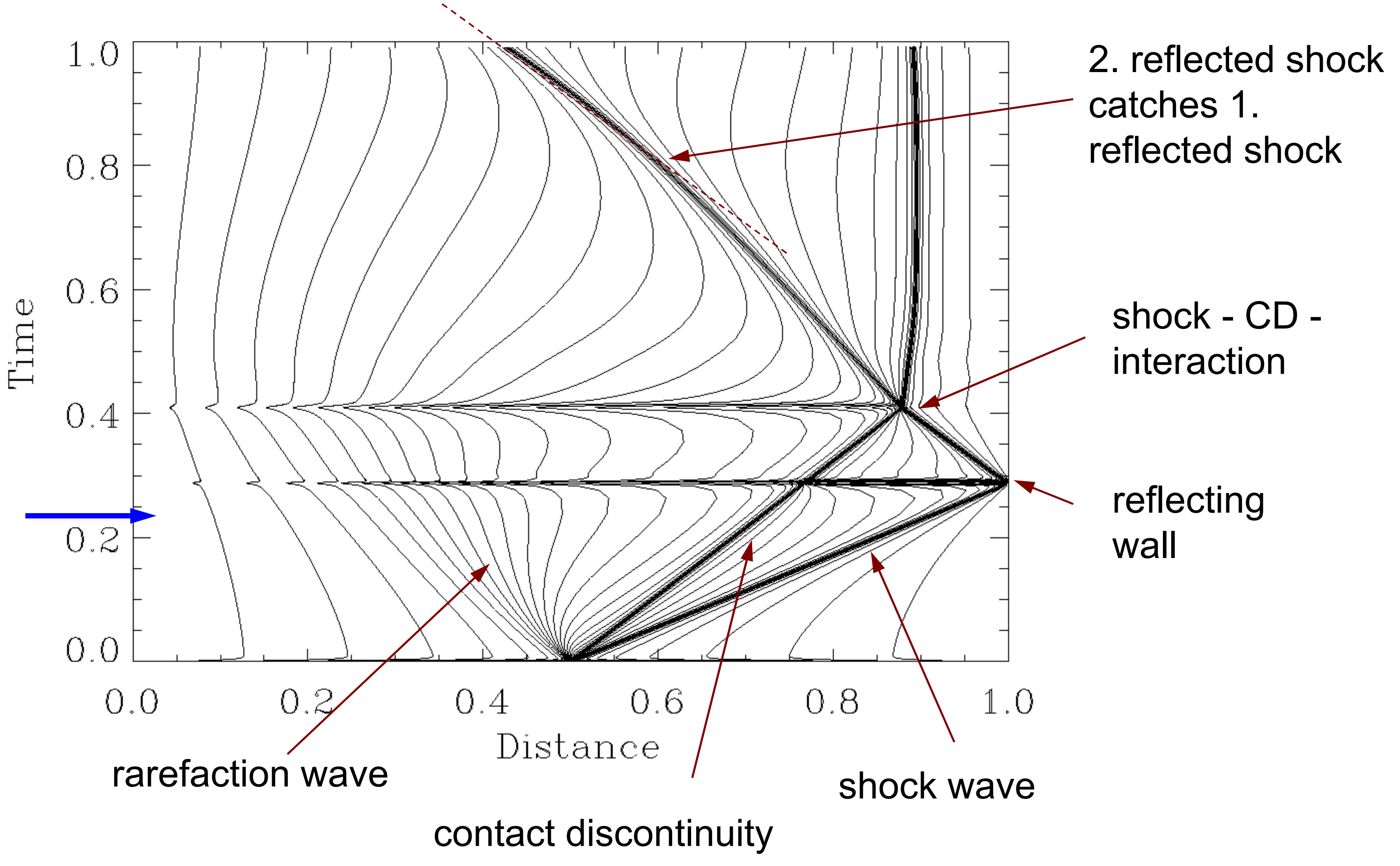
$$\Rightarrow \quad \frac{\tilde{n}_j}{\mathcal{R}_j} = \frac{\tilde{n}_{l-1}}{\mathcal{R}_{j-1}}$$

- Simple and robust equation, controlled by two parameter: α_g, τ_g
- Physical equations are transformed into the moving coordinate system

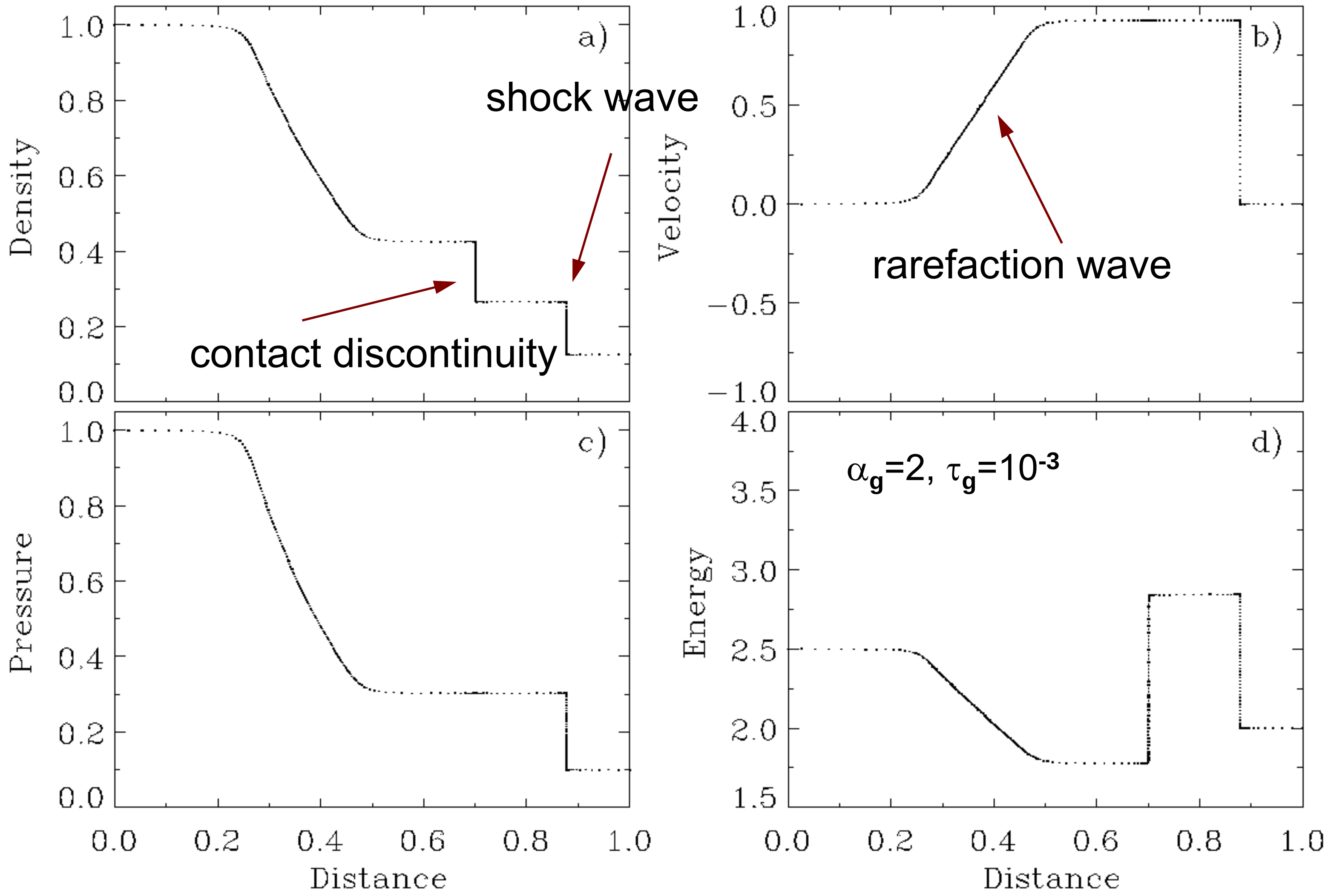
Test grid equation



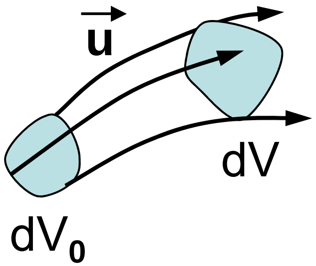
Grid motion for shock tube



Implicit adaptive shock tube



Adaptive Reynolds Theorem



$$dV = J_\mathrm{f}\, dV_0, \quad \frac{D \ln J_\mathrm{f}}{Dt} = \nabla \cdot \mathbf{u}$$

$$dV = J_\mathrm{g}\, dV_0, \quad \frac{D \ln J_\mathrm{g}}{Dt} = \nabla \cdot \mathbf{u}^\mathrm{grid}$$

$$u^{i,\mathrm{rel}} = e^i \cdot (\mathbf{u} - \mathbf{u}^\mathrm{grid})$$

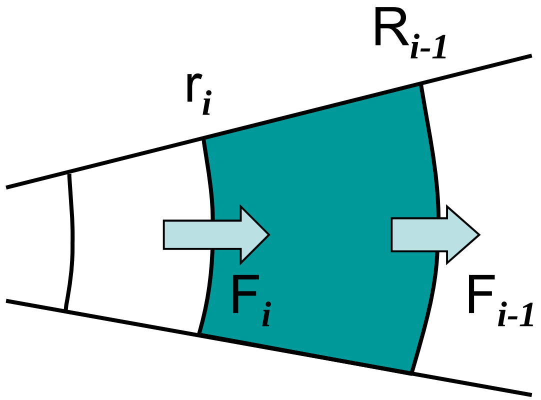
- Transformation of the initial volume element dV_0 through the fluid
- For non-steady coordinate systems (i.e. adaptive grids) the base vectors and the metric are functions of the time
- Variations of the volume element are connected with the derivatives of the velocities in all directions, i.e. even purely radial grids contain the angular contributions
- Covariant relative velocity components u^i are given by projecting them onto the base vectors

Adaptive conservative CFD

$$\frac{\partial}{\partial t}\left[\int_{V(t)} X\, dV\right] + \oint_{\mathcal{O}(t)} X\, \mathbf{u}^{\mathrm{rel}} \cdot d\mathbf{f} = \int_{V(t)} \left[X_{\mathrm{source}} - X_{\mathrm{sink}}\right] dV$$

- Integration over finite but time-dependent volume *V(t)* due to moving grid points
- Advection terms calculated from relative fluxes over cell boundaries
- Relative velocities between mater and grid motion:
  $u^{rel} = e \cdot (u - u^{grid})$

Adaptive conservative CFD

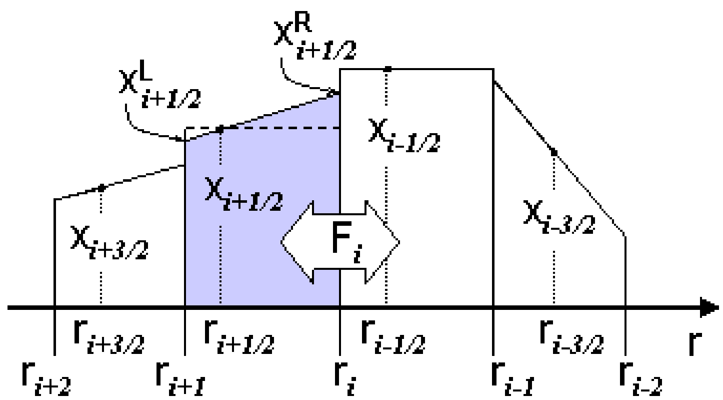


$$F_i = 4\pi\delta t\, r_i^2 u_i^{\mathrm{rel}} \tilde{\rho}_i = \left[4\pi\delta t\, r_i^2 u_i - \frac{4\pi}{3}\delta(r_i^3)\right]\tilde{\rho}_i$$

- Essential property for all modern numerical schemes
- Discrete equations are written in as finite volume, i.e. calculate fluxes F_{i} across cell boundaries to advance integrated quantities
- Here: application to spherical fluxes
- No specialized numerical scheme (e.g. Riemann-solver, etc.) to obtain these fluxes

Advection on staggered meshes

$$r^3_{i+\frac{1}{2}} = \frac{1}{2}\left(r_i^3 + r_{i+1}^3\right)$$



$$x^{\mathrm{L}}_{i+\frac{1}{2}} = x_{i+\frac{1}{2}} - \frac{1}{2}\psi(\theta_{i+\frac{1}{2}})\Delta_i(x)$$

$$x^{\mathrm{R}}_{i+\frac{1}{2}} = x_{i+\frac{1}{2}} + \frac{1}{2}\psi(\theta_{i+\frac{1}{2}})\Delta_i(x)$$

- Transport through moving shells as accurate as possible
- Usage of a staggered mesh, i.e. variables located at cell center or cell boundary
- Fulfil accuracy as well as stability criteria for sub- and supersonic flow
- Avoid numerical oscillations, so-called TVD-schemes
- Ensure correct propagation speed of waves by $\psi(1)=1$

Computational RHD properties

- All variables depend on time and radius, X=X(r,t)
- Equations are discretized in a conservative way, i.e. global quantities are conserved, correct speed of propagating waves
- Adaptive grid to resolve steep features within the flow
- Implicit formulation, large time steps are possible, solution of a non-linear system of equations at every new time step
- Flexible approach to incorporate also new physics

- Complicated Jacobi-matrix has to be derived
- Matrix inversion not always well posed leads to a non-converging iteration

Equations of RHD (1)

$$\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot(\rho\,\vec{u}) = 0$$

- Equation of continuity (conservation of mass)

$$\delta(\rho\, V_{\rm s}) + 4\pi\delta t\,\Delta\left(\widetilde{\rho}\, r^2 u^{\rm rel}\right) = 0$$

- Equation of motion (conservation of linear momentum), including artificial viscosity u_Q

$$\frac{\partial}{\partial t}(\rho\,\vec{u}) + \vec{\nabla}\cdot(\rho\,\vec{u}\,\vec{u}) + \vec{\nabla}P + \rho\vec{\nabla}\phi - \frac{4\pi}{c}\rho\,\kappa_{\rm R}\vec{H} - \vec{u}_{\rm Q} = 0$$

$$\delta(u\,\overline{\rho}\,V_{\rm v}) + 4\pi\delta t\Delta\left((\widetilde{\overline{\rho}u})\,\overline{r^2\,u^{\rm rel}}\right) + 4\pi\,r^2\delta t\,\Delta(P) + \frac{Gm}{r^2}\,\overline{\rho}\,V_{\rm v}\delta t - \frac{4\pi}{c}\overline{\kappa_{\rm R}}\,H\,\overline{\rho}\,V_{\rm v}\delta t + \frac{8\pi}{3r}\delta t\,\Delta\left(\mu_{\rm Q}\,\rho\,\overline{r}^3\left[\frac{\Delta u}{\Delta r} - \frac{\overline{u}}{\overline{r}}\right]\right) = 0$$

Equations of RHD (2)

- Equation of internal gas energy (including artificial viscous energy dissipation ♏$_{\mathrm{Q}}$)

$$\frac{\partial}{\partial t}(\rho\, e) + \vec{\nabla}\cdot(\rho\, e\, \vec{u}) + P\vec{\nabla}\cdot\vec{u} - 4\pi\rho\kappa_{\mathrm{R}}(J-S) - \rho\,\epsilon_{\mathrm{Q}} = 0$$

$$\delta(\rho\, e\, V_{\mathrm{s}}) + 4\pi\delta t\,\Delta\left(\left(\widetilde{\rho e}\right) r^2\, u^{\mathrm{rel}}\right) + 4\pi P\,\Delta(r^2\, u)\delta t$$
$$- 4\pi\kappa_{\mathrm{R}}\,\rho(J-S)V_{\mathrm{s}}\delta t + \frac{2}{3}\mu_{\mathrm{Q}}\,\rho\, V_{\mathrm{s}}\delta t\left(\frac{\Delta u}{\Delta r} - \frac{\overline{u}}{\overline{r}}\right)^2 = 0$$

$$\Delta\phi - 4\pi G\rho = 0$$

$$\Delta m - \rho V_{\mathrm{s}} = 0$$

- Poisson equation leads to gravitational potential, integrated mass m(r) in spherical symmetry

Equations of RHD (3)

- 0^{th} - moment of the RTE, radiation energy density

$$\frac{1}{c}\frac{\partial}{\partial t}J + \frac{1}{c}\vec{\nabla}\cdot(J\vec{u}) + \vec{\nabla}\cdot\vec{H} + \frac{1}{c}K\vec{\nabla}\cdot\vec{u} - \frac{1}{c}\frac{3K-J}{r}\vec{u} + \rho\,\kappa_{\mathrm{R}}(J-S) = 0$$

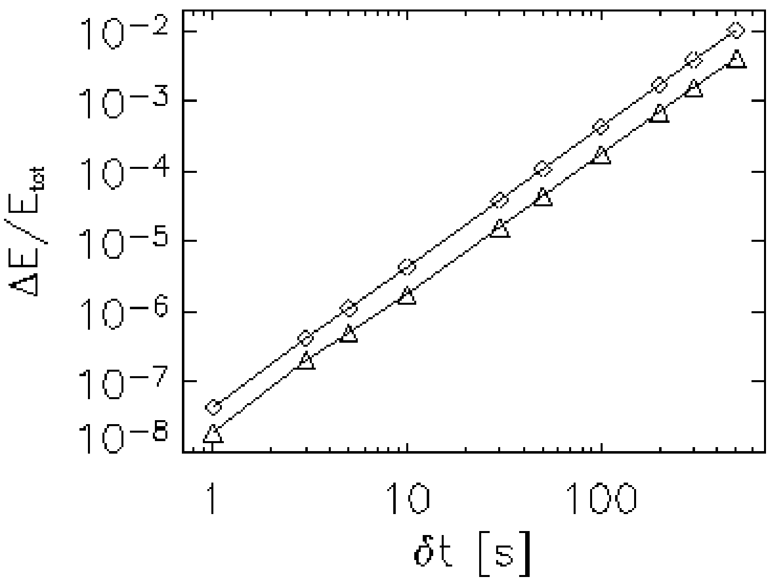
$$\frac{1}{c}\delta(J\,V_{\mathrm{s}}) + \frac{4\pi}{c}\delta t\Delta\left(r^2\,u^{\mathrm{rel}}\widetilde{J}\right) + 4\pi\delta t\Delta(r^2\,H) + \frac{4\pi}{c}\delta t K\,\Delta(r^2\,u) - \frac{1}{c}\delta t\,(3K-J)\,\frac{\overline{u}}{\overline{r}}V_{\mathrm{s}} + \kappa_{\mathrm{R}}\rho\,\delta t\,(J-S)\;V_{\mathrm{s}} = 0$$

- 1^{th}- moment of RTE, equation of radiative flux

$$\frac{1}{c}\frac{\partial}{\partial t}\vec{H} + \frac{1}{c}\vec{\nabla}(\vec{H}\cdot\vec{u}) + \vec{\nabla}K + \frac{3K-J}{r} + \frac{1}{c}\vec{H}\vec{\nabla}\cdot\vec{u} + \rho\,\kappa_{\mathrm{R}}\vec{H} = 0$$

$$\frac{1}{c}\delta\,(H\,V_{\mathrm{v}}) + \frac{4\pi}{c}\delta t\,\Delta\left(\overline{r^2}\,\widetilde{H}\,\overline{u^{\mathrm{rel}}}\right) + 4\pi r^2\delta t\,\Delta(K) + \delta t\frac{3\overline{K}-\overline{J}}{r}V_{\mathrm{v}} + \frac{4\pi}{c}\delta t\,H\,\Delta\left(\overline{r^2}\,\overline{u}\right) + \overline{\kappa_{\mathrm{R}}}\,\delta t\,H\,\overline{\rho}\,V_{\mathrm{v}} = 0$$

Temporal discretization



$$\frac{\partial \vec{x}}{\partial t} = H(\vec{x})$$

- 2^{nd}-order temporal discretization to reduce artificial damping of oscillations
- Smallest errors in case of time-centred variables, i.e. $x = \theta\, x^{(n+1)} + (1-\theta)\, x^{(n)}$

$$\frac{\vec{x}^{(n+1)} - \vec{x}^{(n)}}{\delta t} = (1-\theta)\, H(\vec{x}^{(n)}) + \theta\, H(\vec{x}^{(n+1)})$$

$$\frac{\vec{x}^{(n+1)} - \vec{x}^{(n)}}{\delta t} = H(\bar{\vec{x}}) = H((1-\theta)\vec{x}^{(n)} + \theta \vec{x}^{(n+1)})$$

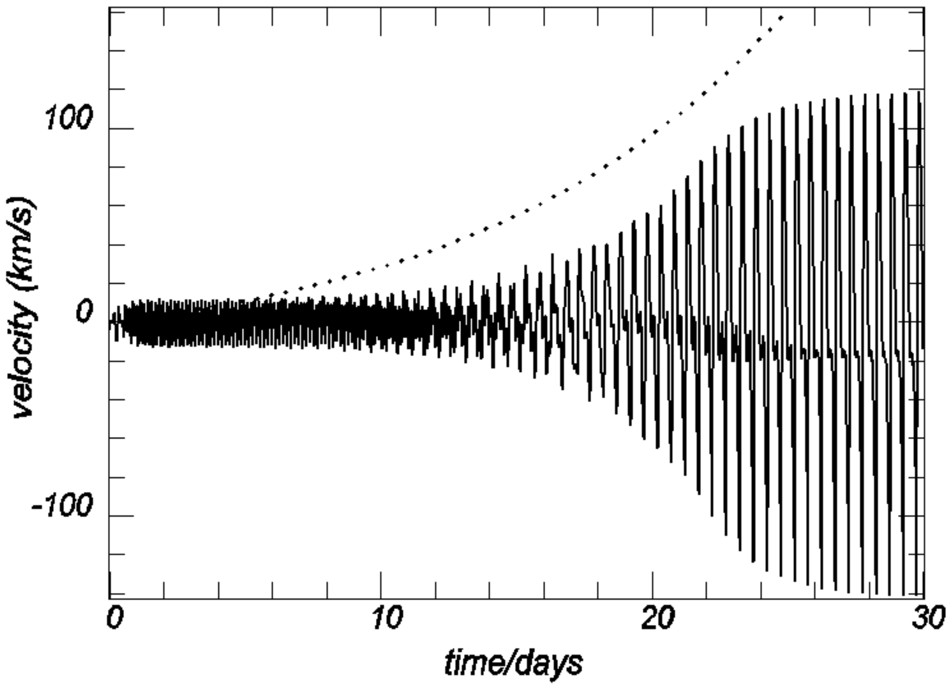
Examples and Applications

- Radiation Hydrodynamics: 7 Equations for density, gas velocity, gas internal energy, radiation energy density, radiation flux, radius)
  - stellar pulsations
  - star and giant planet formation
  - Radiation driven winds
- Dusty radiation hydrodynamics: 12 Equations, RHD + dust formation through moments
  - Dust driven winds
- Cosmic ray hydrodynamics: 6 Equations, HD + 2 energy equations for cosmic rays and magnetosonic waves
  - Evolution of SNR including acceleration of energetic particles
  - Galactic winds

RHD Requirements

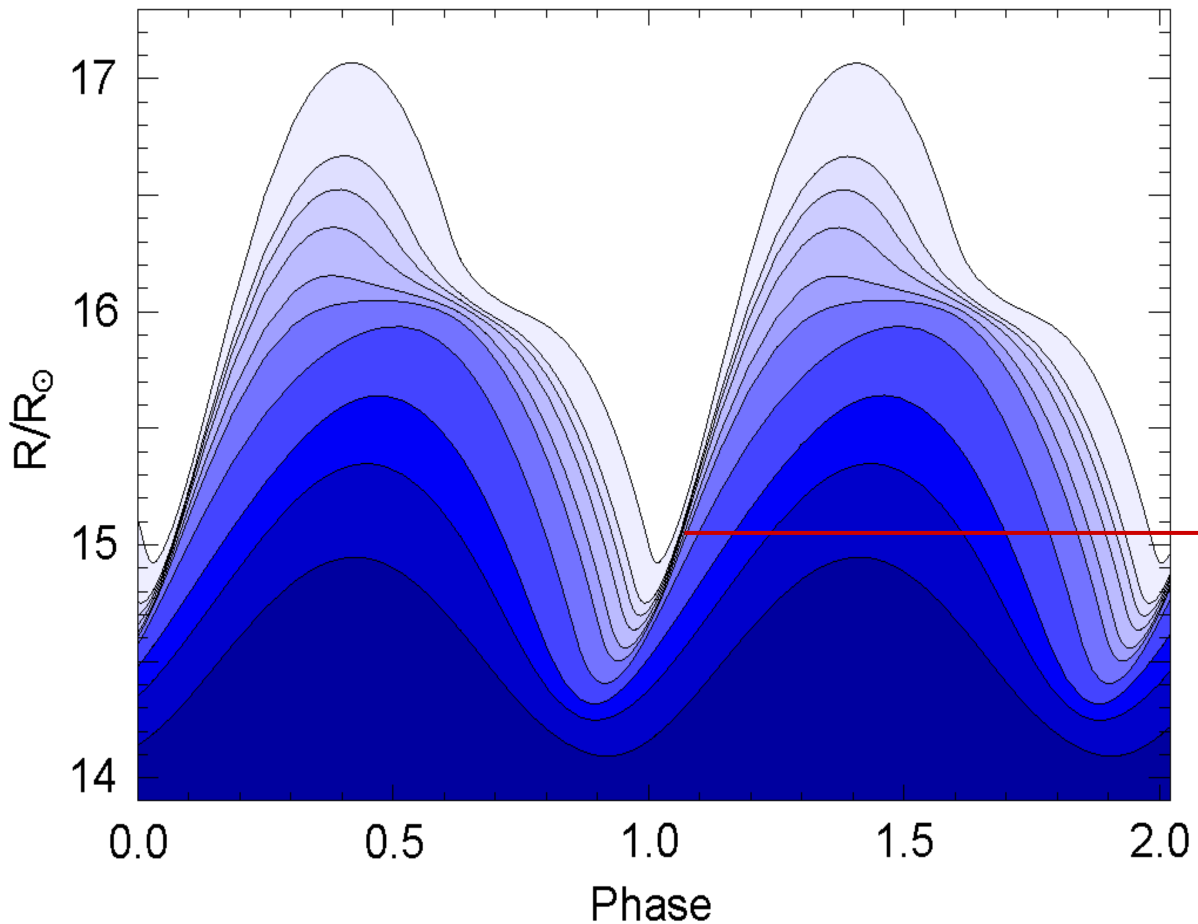
- Resolve relevant features within one single computation like driving zone, ionization zones, opacity changes, shock waves, stellar winds, …
- Kinetic energy is only small fraction of the total energy of the pulsation star
- Steep gradients within the stellar atmosphere and/or possible changes of the atmospheric stratification due to energy deposition may change boundary conditions
- Long term evolution of stellar pulsations, secular changes on thermal time scales, i.e. $\mathbf{t_{KH} >> t_{dyn}}$
- Solve full set of **Radiation Hydrodynamics** (RHD), problem: detailed properties of **convection**

Growth of pulsations



- Pulsations initiated by a small random perturbation: 5 km/s
- Initial linear growth (dotted line), stellar atmosphere can adjust on a different time scale
- Final amplitude when kinetic energy becomes constant
- Model **WR123U:** $M=25\,M_{\odot}$, $T_{eff}=33\,900$ K, $L=2.82 \cdot 10^{5} L_{\odot}$

Atmosphere with shock waves



$M = 25\,M_\odot$

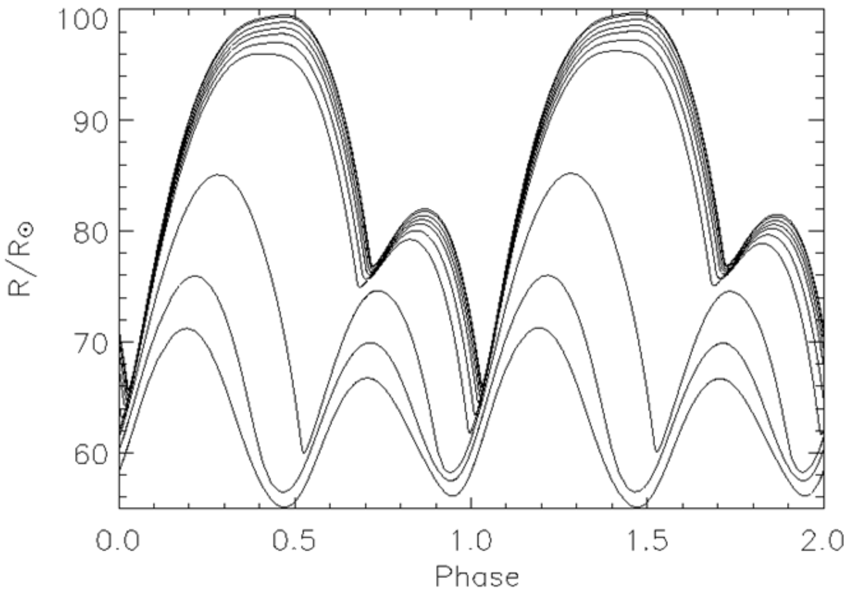
$L = 282\,000\,L_\odot$

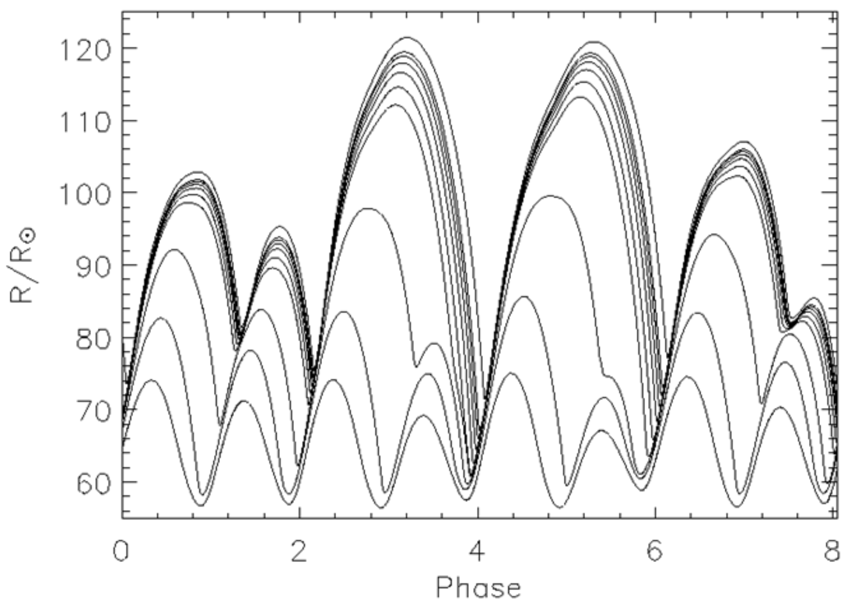
$T_{eff} = 33\,900\,K$

$P = 0.49$ days

Shock wave

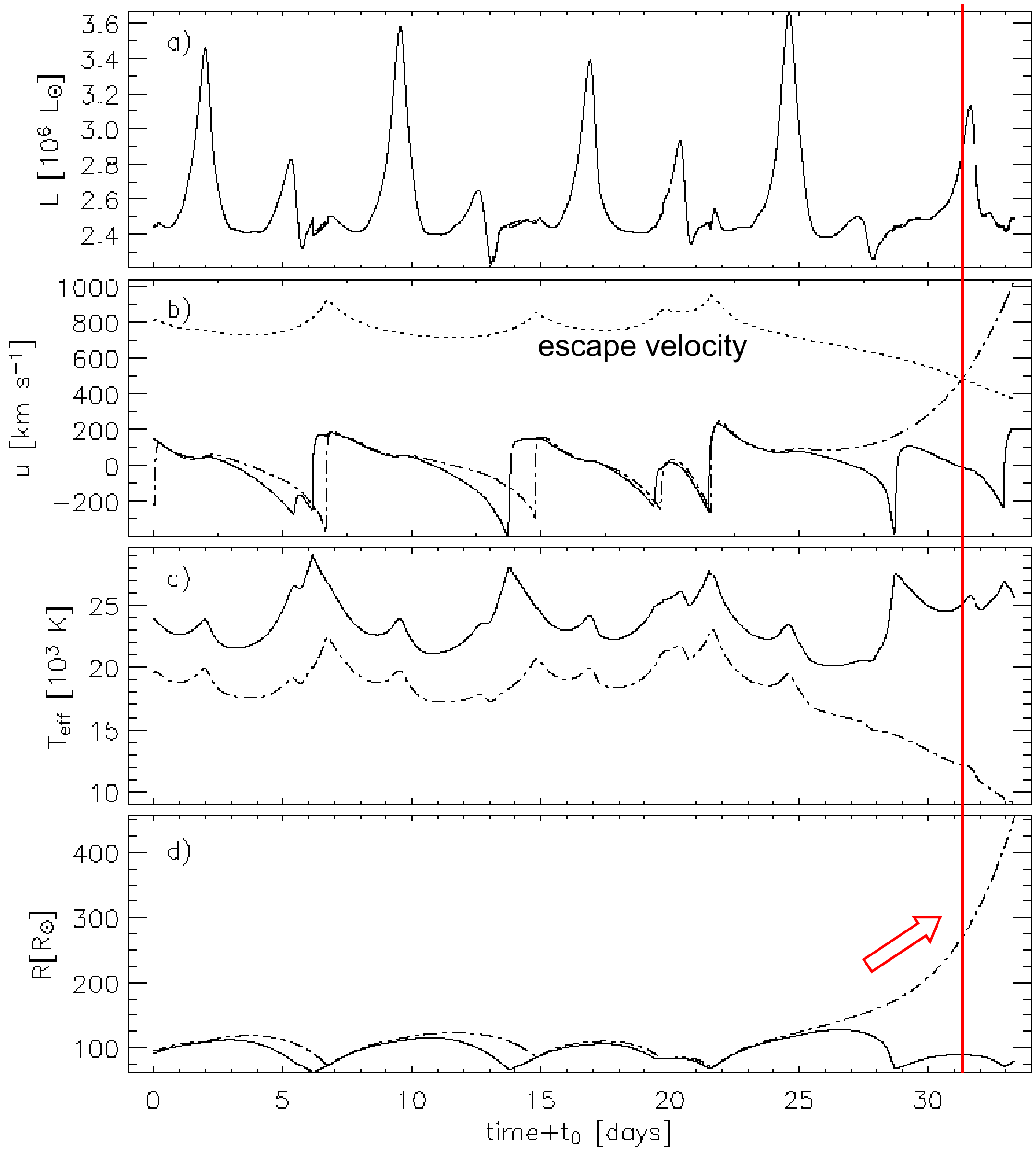
Ballistic motions on the scale of t_{ff}





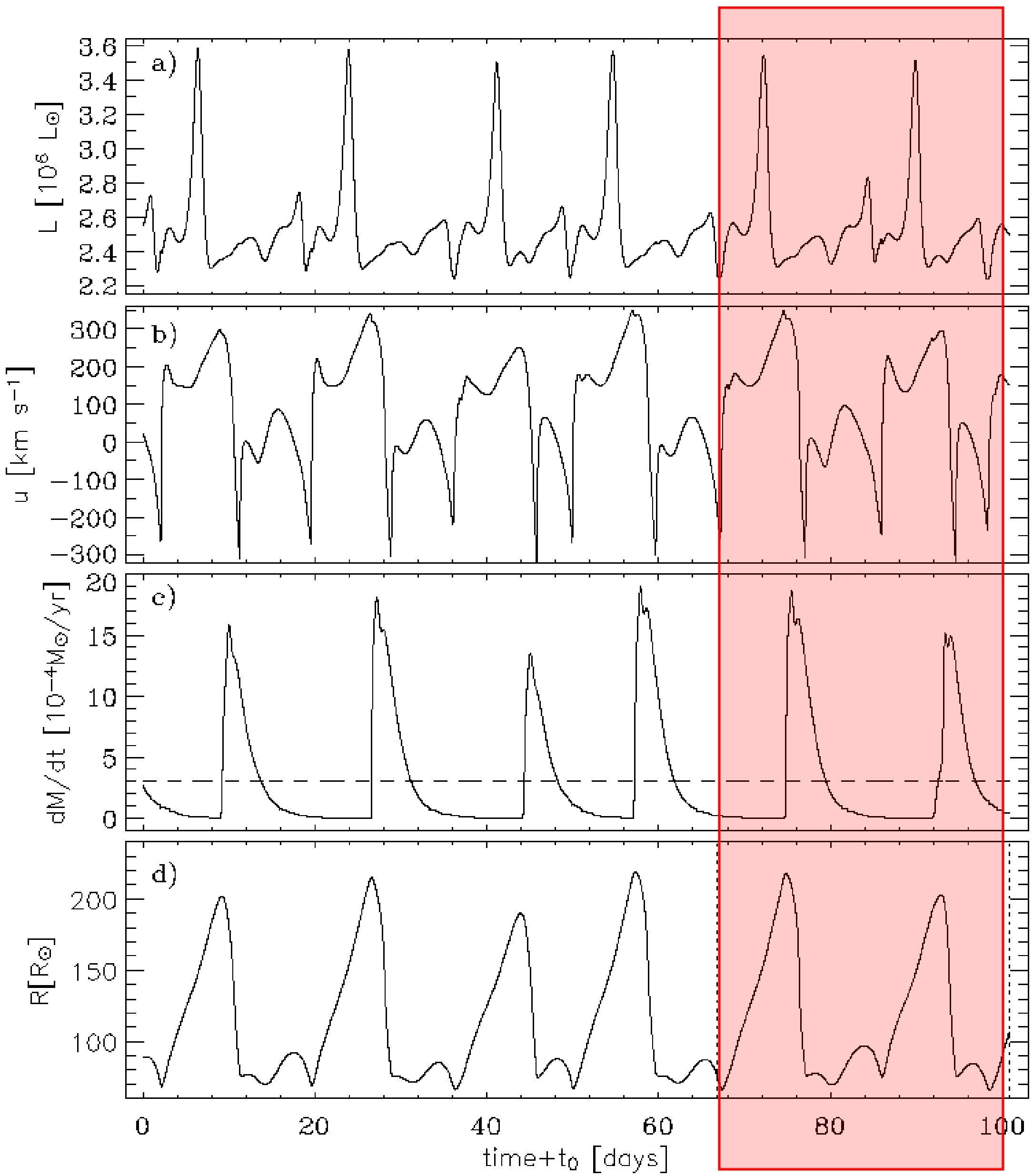
Atmospheric dynamics

- *IRS16WS* model: L=2.59•10^{6}L$_\odot$
- Rotation plays important role in decoupling the stellar atmosphere from internal pulsations
- Ballistic motions at different time scales introduce complex flows
- v_{rot}=220 km/s, P=3.471d, T=25000K
- v_{rot}=225 km/s, P=3.728d, T=24000K
- Higher rotation rates lead to mass loss of about 10^{-4} M$_\odot$/yr



Initiating mass loss

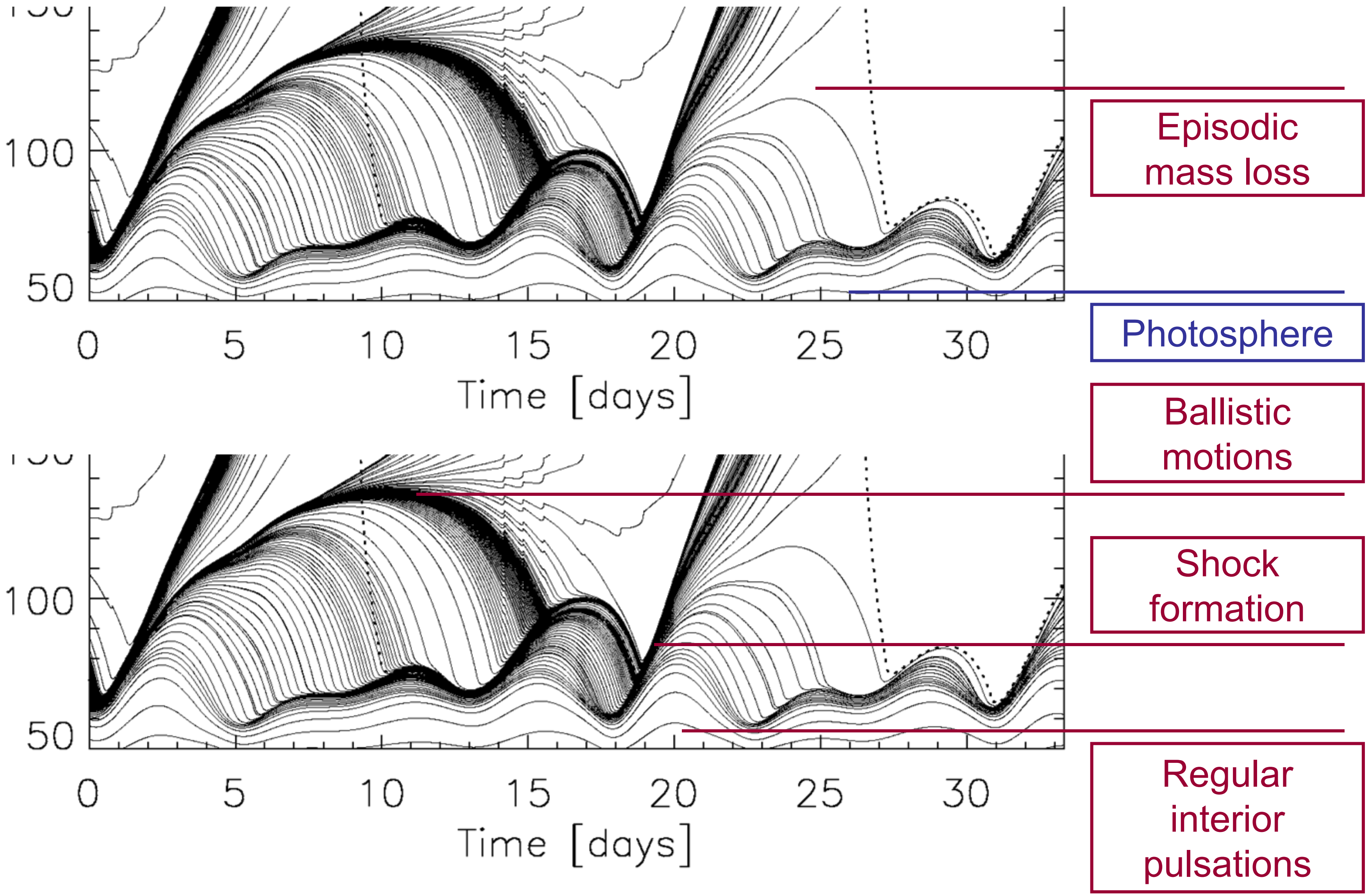
- Pulsation perturbed by increase rotational velocity from 225km/s to 230 km/s
- After 4 cycles outermost mass shell accelerated beyond escape velocity
- Outer boundary: from Lagrangian to outflow at 400 $R_{\odot}$, advantage of adaptive grid
- Gas velocity varies there around 550 km/s



Pulsation and mass loss

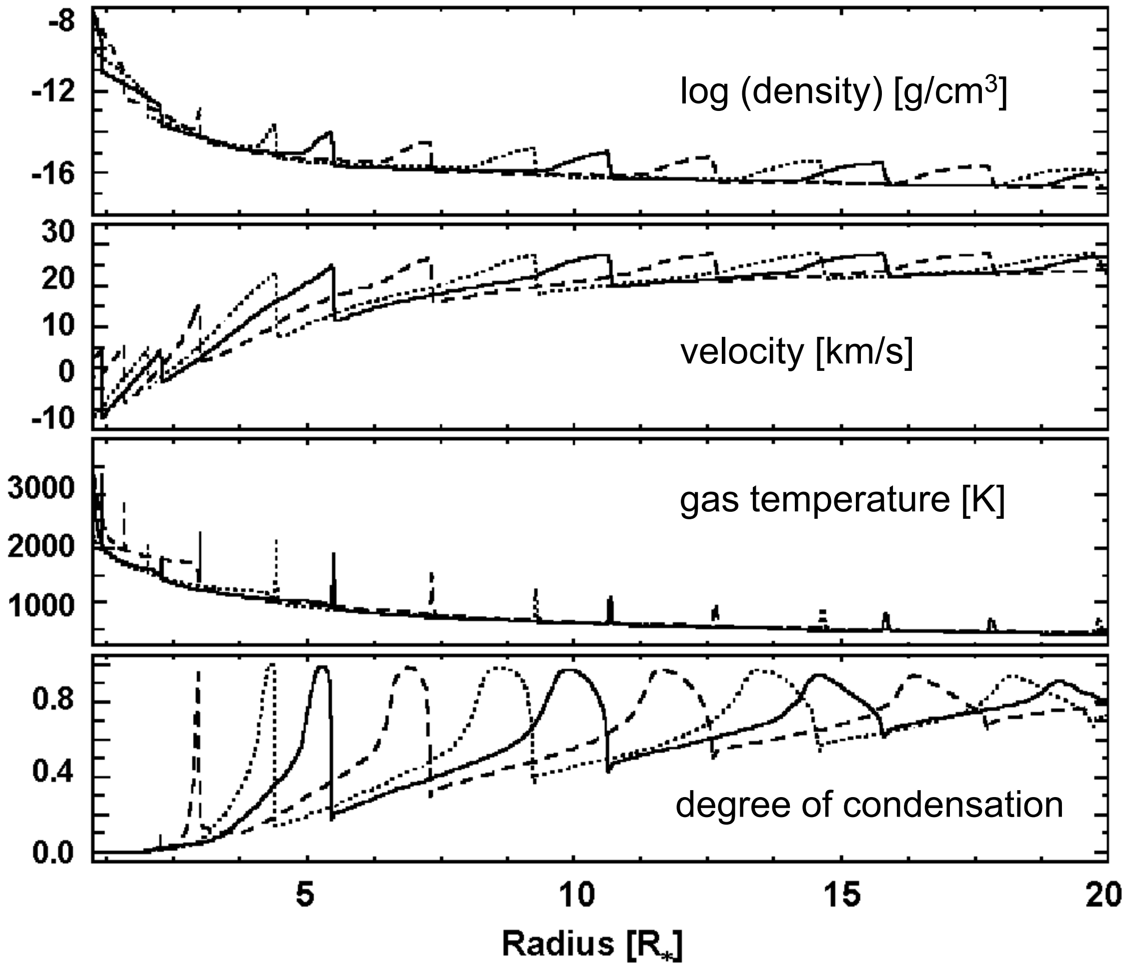
- Pulsation still exists, very different outer boundary condition
- Large photospherical velocity variations due to changes in the optical depth
- Mean equatorial mass loss: $3 \cdot 10^{-4} M_{\odot}/yr$, v_{ext}=550 km/s
- Total mass loss rate probable reduced by angle-dependence

Motion of mass shells



Dust driven winds

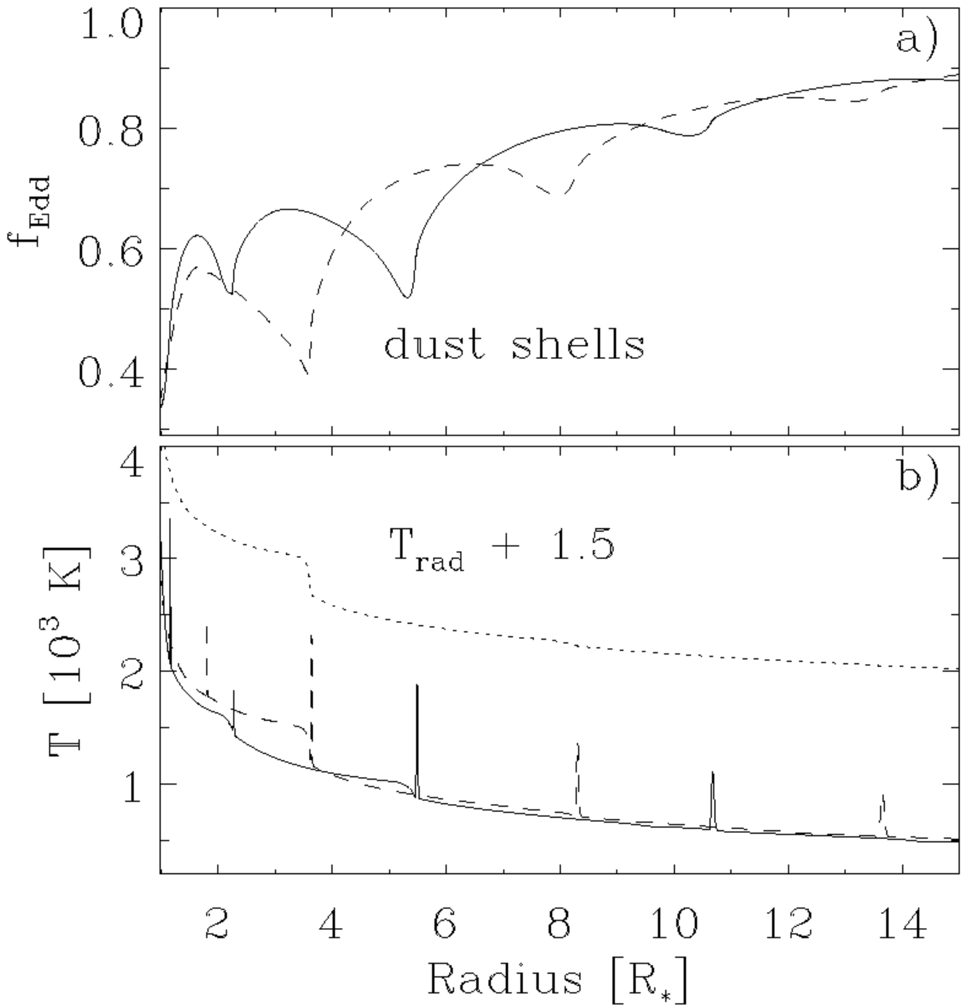
PISTON



Red Giant: v = 25 km/s, $\dot{M} = 2 \cdot 10^{-6}\ M_\odot$/yr

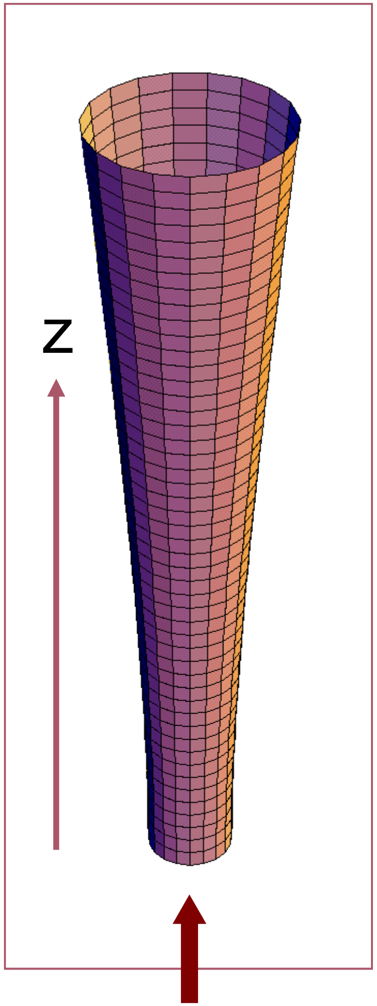
- Freely moving grid in one dimension
- Clustering around steep gradients, smoothing
- Trace the moving shock fronts
- Supercritical shock waves, gas compression and radiatively cooled, relaxation zones
- Dust formation in cooling zones, accelerating the flow
- Solutions are more stationary on the moving grid

Supercritical shocks



- Smooth radiation field across shock wave, gas temperature jumps
- Thermal gas relaxes towards up-stream temperature
- Necessary to resolve this feature
- Backwarming effect at large opacities
- Isotropy of radiation field due to opacity increase by dust formation

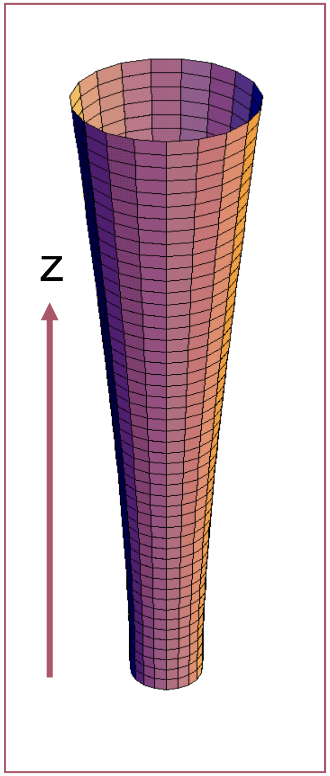
Physical Model



Time-dependent boundary conditions

- Galactic winds are driven by the collective effects and the interaction of thermal pressure, cosmic rays and Alfvén-waves (details e.g. Breitschwerdt et al.1991)
- Flux-tube geometry, variables depend on distance z and time t, area A(z) is specified
- Various sources of time-dependence, shock waves, particle acceleration, wave-heating and damping, diffusion, radiative losses, ...
- Numerical solutions on an 1-dimensional adaptive grid to resolve flow features

Initial and boundary conditions



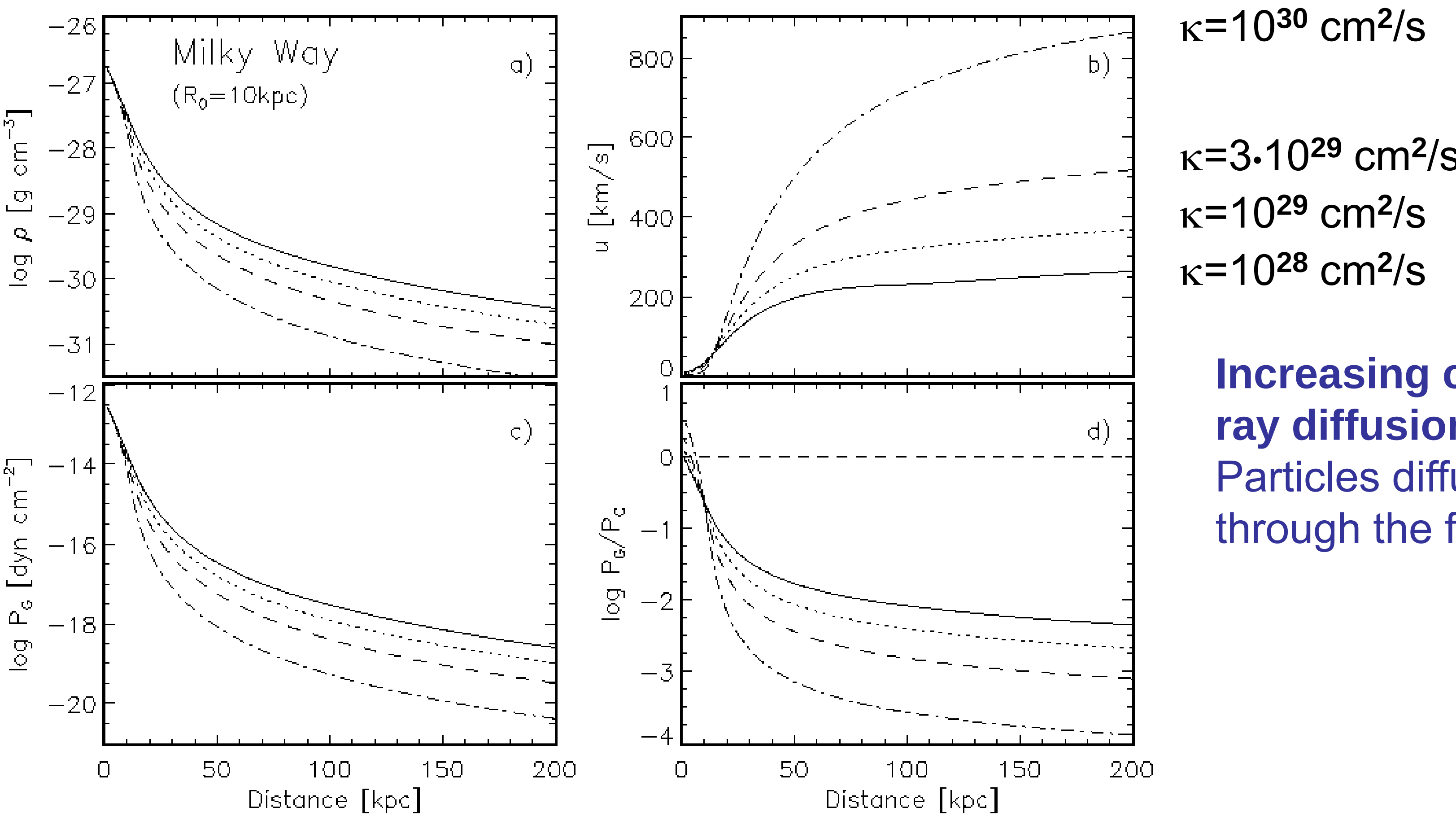
- Stationary winds: perturbation of boundary conditions, temporal development towards another stationary solution
- Density, gas pressure, wave pressure and cosmic ray flux F_c given at inner boundary located e.g. at z=100pc above the galactic plane:

$$F_c = \frac{\gamma_c}{\gamma_c - 1}(u + v_A)P_c - \frac{\bar{\kappa}}{\gamma_c - 1}\frac{\partial P_c}{\partial z}$$

- Velocity through critical point determined
- Flux-tube geometry:

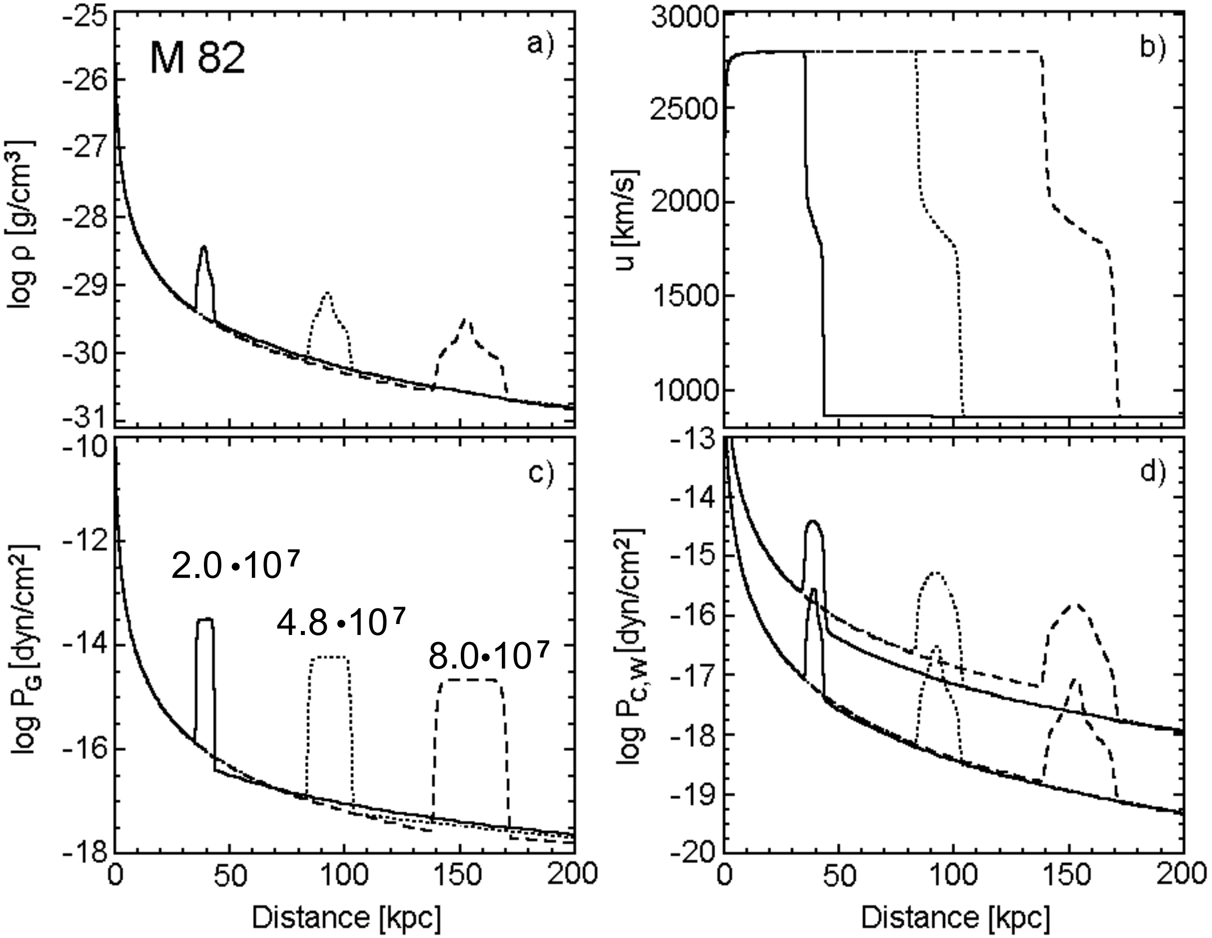
$$A(z) = A_0\left[1 + \left(\frac{z}{z_0}\right)^2\right], \quad z_0 = 100\,\mathrm{pc}$$

Effects of particle diffusion



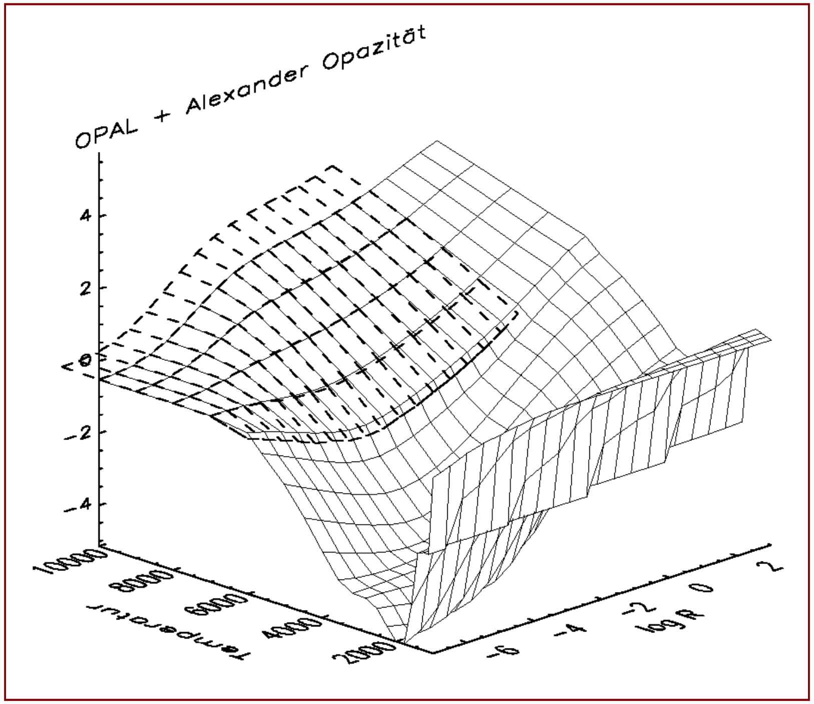
Increasing cosmic ray diffusion:
Particles diffuse through the flow

Shocks in Galactic winds



- Inner boundary: $\Delta t=10^6$ yr to increase of gas and CR by factor 10
- Two shock propagate within the wind
- At the shocks CR are reaccelerated
- Solutions become stationary when shock left computational domain

Interpolation by rational splines



- Smooth data also between computed points (opacities, EOS, chemical constants, cooling curves, nuclear reaction rates, …)
- Interpolation together with smooth derivatives
- Rational splines are recommended

$$\mathcal{F}_{ij}(x,y) = \sum_{k=1}^{4}\sum_{l=1}^{4} a_{ijkl}\, g_k(x)\, g_l(y)$$

$$g_k(x) = a_{1k}\, t + a_{2k}\, s + a_{3k}\frac{t^3}{1+qs} + a_{4k}\frac{s^3}{1+qt}$$

$$\text{and} \quad t = \frac{x - x_i}{x_{i+1} - x_i}, \quad s = 1 - t, \quad x_i \le x \le x_{i+1}$$

1D to 2D grid equation

$$n_j = \frac{\mathcal{X}_j}{x_j - x_{j+1}}$$

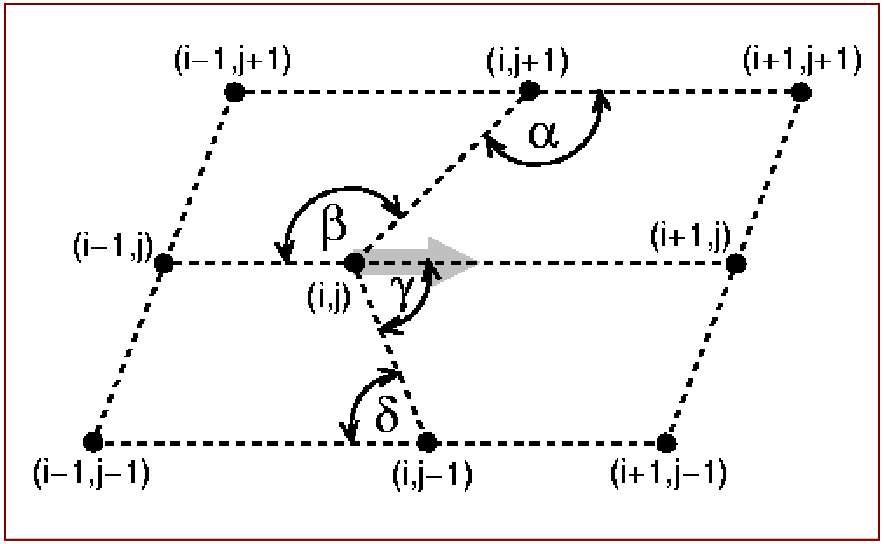
$$n_\xi = \frac{1}{|\Delta_\xi \vec{x}|} = \frac{1}{\sqrt{(\Delta_\xi x)^2 + (\Delta_\xi y)^2}} = \frac{1}{\sqrt{(x_{i+1,j} - x_{i,j})^2 + (y_{i+1,j} - y_{i,j})^2}}$$

$$\mathcal{R} \propto \sqrt{1 + \left(\frac{df}{dx}\right)^2}$$

$$R_\xi = \sqrt{1 + \sum_k \alpha^k \left(\frac{\Delta_\xi u^k}{\Delta_\xi \vec{x}}\right)^2} = \sqrt{1 + \sum_k \alpha^k \frac{\left(u^k_{i+1,j} - u^k_{i,j}\right)^2}{(x_{i+1,j} - x_{i,j})^2 + (y_{i+1,j} - y_{i,j})^2}}$$

$$\frac{\tilde{n}_j}{\mathcal{R}_j} = \frac{\tilde{n}_{l-1}}{\mathcal{R}_{j-1}} \quad \rightarrow \quad R_\xi / n_\xi \Big|_{i-1,j} = R_\xi / n_\xi \Big|_{i,j}$$

- Basic concept remains: resolution and point concentration along coordinate lines (η,ξ)
- Spatial and temporal smoothing
- Additional features due to skewness control and regularity
- No final solution, but see e.g. Stökl and Dorfi (2007)

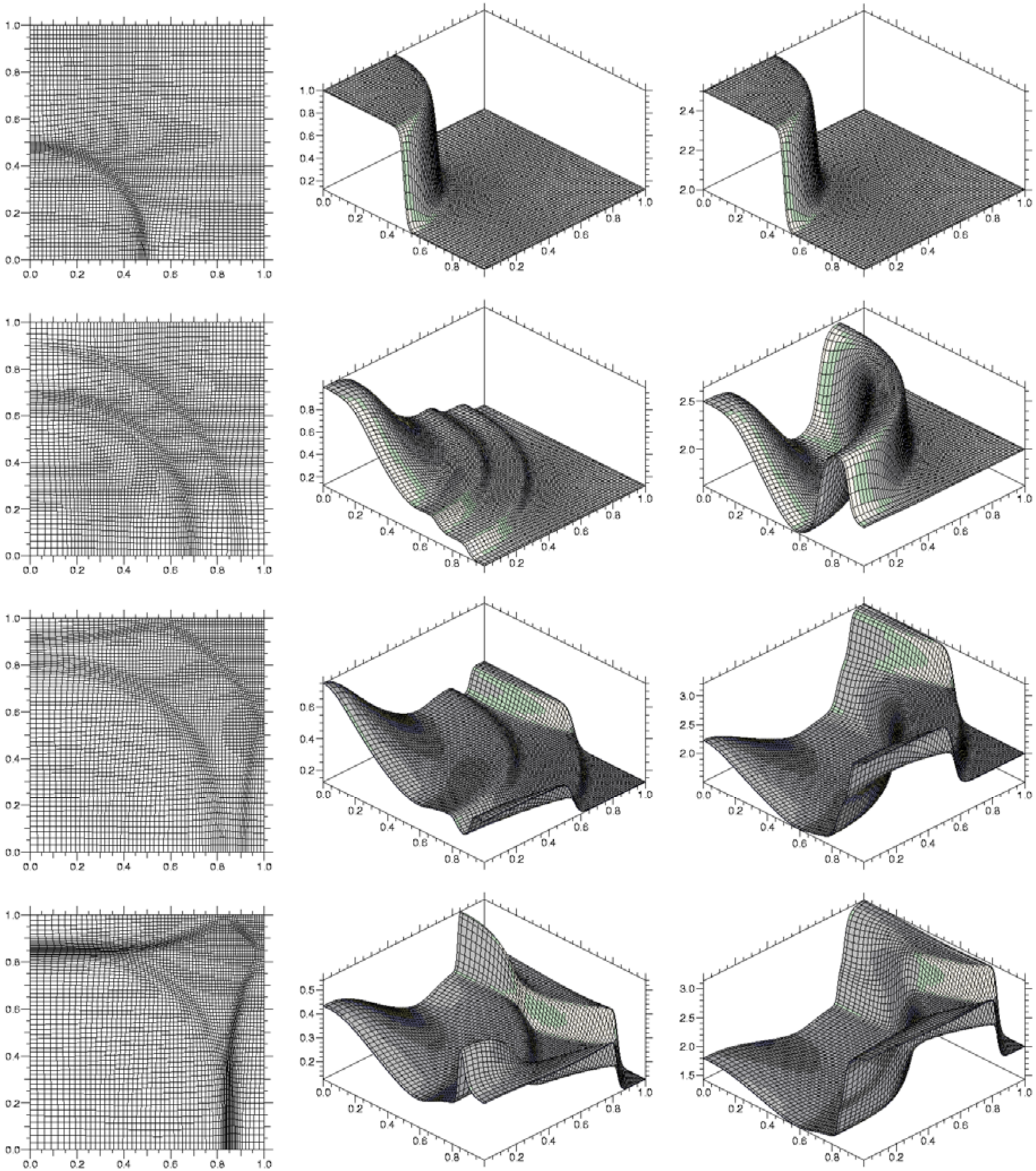


2D skewness

$$
\begin{aligned}
\mathrm{IP}_\xi^\alpha &= \left(\vec{x}^{\,\mathrm{m}}_{i,j+1} - \vec{x}^{\,\mathrm{a}}_{i,j}\right) \cdot \left(\vec{x}^{\,\mathrm{m}}_{i+1,j+1} - \vec{x}^{\,\mathrm{m}}_{i-1,j+1}\right) \\
\mathrm{IP}_\xi^\beta &= \left(\vec{x}^{\,\mathrm{m}}_{i,j+1} - \vec{x}^{\,\mathrm{a}}_{i,j}\right) \cdot \left(\vec{x}^{\,\mathrm{m}}_{i+1,j} - \vec{x}^{\,\mathrm{m}}_{i-1,j}\right) \\
\mathrm{IP}_\xi^\gamma &= \left(\vec{x}^{\,\mathrm{a}}_{i,j} - \vec{x}^{\,\mathrm{m}}_{i,j-1}\right) \cdot \left(\vec{x}^{\,\mathrm{m}}_{i+1,j} - \vec{x}^{\,\mathrm{m}}_{i-1,j}\right) \\
\mathrm{IP}_\xi^\delta &= \left(\vec{x}^{\,\mathrm{a}}_{i,j} - \vec{x}^{\,\mathrm{m}}_{i,j-1}\right) \cdot \left(\vec{x}^{\,\mathrm{m}}_{i+1,j-1} - \vec{x}^{\,\mathrm{m}}_{i-1,j-1}\right)
\end{aligned}
$$

- Basic concept remains: coordinate lines (η,ξ)
- Skewness control and regularity not jet clear
- Important role of boundary conditions
- Up to now only for Cartesian geometries, φ=0.5

$$
R_\xi/n_\xi^{\mathrm{s}}\Big|_{i,j} - R_\xi/n_\xi^{\mathrm{s}}\Big|_{i-1,j} - \frac{1}{\Delta\eta}\Big[\varphi^{\perp}(\mathrm{IP}_\xi^\delta - \mathrm{IP}_\xi^\alpha) + \varphi^{|}(\mathrm{IP}_\xi^\gamma - \mathrm{IP}_\xi^\beta)\Big] = 0
$$



2D-HD

- 2-dimensional adaptive, implicit calculations based on the same numerical methods
- Example: Spherical shock tube on Cartesian grid
- 70x70 grid points, donor-cell advection scheme

Stökl & Dorfi 2007

Computational Remarks

- Never accept the computed results without a critical study
- Start with simple problems, compare these results to analytical solutions or results from ODE computations
- Perform grid refinement if possible, convergence of results

- Non-conservative schemes lead to wrong propagation speeds
- Be careful with stiff source terms
- Smoothing profiles by artificial viscosity needs special treatments
- Evaluation of start-up errors
- Watch grid orientation effects, transport goes preferentially along grid lines

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